

On the spectrum of sizes of semiovals contained in the Hermitian curve

Daniele Bartoli*, György Kiss†,
Stefano Marcugini‡ and Fernanda Pambianco‡

Abstract

Some constructions and bounds on the sizes of semiovals contained in the Hermitian curve are given. A construction of an infinite family of 2-blocking sets of the Hermitian curve is also presented.

1 Introduction

Let Π_q be a finite projective plane of order q and let $\text{PG}(2, q)$ denote the Desarguesian projective plane over the finite field of q elements, \mathbb{F}_q . A *semioval* \mathcal{S} in Π_q is a non-empty pointset with the property that for every point $P \in \mathcal{S}$ there exists a unique line t_P such that $\mathcal{S} \cap t_P = \{P\}$. This line is called the tangent line to \mathcal{S} at P . A *blocking semioval* is a semioval \mathcal{S} such that each line of $\text{PG}(2, q)$ contains at least one point of \mathcal{S} and at least one point outside \mathcal{S} . A blocking semioval existing in every projective plane of order $q > 2$ is the vertexless triangle, the set of points formed by the union of three non-concurrent lines with the points of intersections removed.

The classical examples of semiovals arise from polarities (ovals and unitals), and from the theory of blocking sets. The semiovals are interesting objects in their own right, but the study of semiovals is also motivated by their applications to cryptography. Batten constructed in [4] an effective message sending scenario which uses determining sets. She showed that blocking semiovals are a particular type of determining sets in projective planes.

*The author acknowledges the support of the European Community under a Marie-Curie Intra-European Fellowship (FACE project: number 626511).

†The research was supported by the Hungarian National Foundation for Scientific Research, Grant NN 114614.

‡The research was supported by the Italian MIUR (progetto 40% “Strutture Geometriche, Combinatoria e loro Applicazioni”), and by GNSAGA (INDAM).

It is known that if \mathcal{S} is a semioval in Π_q then $q + 1 \leq |\mathcal{S}| \leq q\sqrt{q} + 1$ and both bounds are sharp [19, 28]; the extremes occur when \mathcal{S} is an oval or a unital. In particular in $\text{PG}(2, q)$ a conic has $q + 1$ points and if q is a square then a Hermitian curve has $q\sqrt{q} + 1$ points. A survey on results about semiovals can be found in [20].

In the last years the interest and research on the fundamental problem of determining the spectrum of the values for which there exists a given subconfiguration of points in $\text{PG}(n, q)$ have increased considerably (see for example [1–3, 5, 7, 9, 18, 21, 23–25]). For $q \leq 9$, q odd, the spectrum of sizes of semiovals was determined by Lisonek [22] by exhaustive computer search. Kiss, Marcugini, and Pambianco [21] extended Lisonek’s results to the cases $q = 11$ and 13.

There are many known constructions and theoretical results about semiovals, in particular those that either contain large collinear subsets in which case their size is close to the lower bound, or their size is close to the upper bound. In the latter case Kiss, Marcugini and Pambianco [21] constructed semiovals by careful deletion of points from a unital. If q is an odd square then they gave explicit examples of semiovals of size k for all k satisfying the inequalities $q(\sqrt{q} + 1)/2 \leq k \leq q\sqrt{q} + 1$ and they proved existence if $q(\lceil 4 \log q \rceil + 1) \leq k \leq q\sqrt{q} + 1$ holds. The unital they started from was originally constructed by Szőnyi [27]. It is a pencil of superosculating conics which is also a minimal blocking set. Later on, Dover and Mellinger gave the complete characterization of semiovals from unions of conics [12].

Our goal in this paper is to give similar constructions and estimates on the sizes of semiovals coming from the classical unital, the Hermitian curve. The main result is an explicit construction of semiovals of size k for all k satisfying the inequalities $2q\sqrt[4]{q} + 4q - 2\sqrt[4]{q^3} - 3\sqrt{q} + 1 \leq k \leq q\sqrt{q} + 1$ if $q = s^4$ and odd (see Corollary 2.9). We also present explicit examples for other values of q and prove the existence of semiovals of size k for q odd if $(q - \sqrt{q} + 1) \left\lceil \frac{4(\sqrt{q}+1)}{\sqrt{q}-1} \log q \right\rceil \leq k \leq q\sqrt{q} + 1$. If q is large enough, then this gives a slight improvement on the previously known bound for almost all q . Our main tools are the application of proper blocking sets of the Hermitian curve constructed by Blokhuis et al. [6] and the decomposition of the Hermitian curve into a union of $(q - \sqrt{q} + 1)$ -arcs, originally given by Seib [26], see in English in [14].

Finally, in the last section, we present a construction of an infinite family of 2-blocking sets of the Hermitian curve.

2 Explicit constructions of semiovals

For the sake of convenience from now on we work on planes of order q^2 . In this section we construct various examples of semiovals in $\text{PG}(2, q^2)$ arising from the points of the Hermitian curve \mathcal{H}_q . This curve has $q^3 + 1$ points, there is a unique tangent line to \mathcal{H}_q at each of its points and each of the

Table 1: Semiovals contained in the Hermitian curve of $\text{PG}(2, 9)$

Sizes	12	15	16	18	19	20	21	22	23	24	25	26	27	28
Non-equivalent examples	1	1	2	5	4	9	10	5	8	6	2	1	1	1

other $q^4 - q^3 + q^2$ lines of $\text{PG}(2, q^2)$ is a $(q+1)$ -secant of \mathcal{H}_q . A pointset $\mathcal{D} \subset \mathcal{H}_q$ is called a *2-blocking set* if each $(q+1)$ -secant contains at least 2 points of \mathcal{D} . For a detailed description of \mathcal{H}_q we refer to [16].

For $q = 2, 3$ we could perform exhaustive computer search and the situation is the following. In $\text{PG}(2, 4)$, semiovals contained in the Hermitian curve exist only of sizes 6, 8, 9: this means that Theorem 2.3 gives the complete spectrum of semiovals contained in the Hermitian curve for $q = 2$. The spectrum of the sizes of semiovals contained in the Hermitian curve of $\text{PG}(2, 9)$ and the number of non-equivalent examples (up to collineations) are presented in Table 1.

For the constructions we need the following elementary observations.

Proposition 2.1. *Let \mathcal{S} be a semioval. Suppose that the pointset $\mathcal{T} \subset \mathcal{S}$ has the property that if ℓ is a secant line to \mathcal{S} then the inequality $|\mathcal{S} \cap \ell| \geq |\mathcal{T} \cap \ell| + 2$ holds. Then $\mathcal{S} \setminus \mathcal{T}$ is a semioval and \mathcal{S} contains semiovals of size k for all k satisfying the inequalities $|\mathcal{S} \setminus \mathcal{T}| \leq k \leq \mathcal{S}$.*

Proof. Let R be a point of $\mathcal{S} \setminus \mathcal{T}$. Then the tangent to \mathcal{S} at R is obviously a tangent to $\mathcal{S} \setminus \mathcal{T}$ at R . We have to prove that no new tangents appear after the deletion of points of \mathcal{T} . But if a line ℓ meets \mathcal{S} in more than one point, then

$$|(\mathcal{S} \setminus \mathcal{T}) \cap \ell| = |\mathcal{S} \cap \ell| - |\mathcal{T} \cap \ell| \geq 2.$$

Thus no former secant line becomes tangent line to $\mathcal{S} \setminus \mathcal{T}$, so it is a semioval.

Let $|\mathcal{S} \setminus \mathcal{T}| = k_0$. If \mathcal{U} is any subset of $k - k_0$ points of \mathcal{T} then $((\mathcal{S} \setminus \mathcal{T}) \cup \mathcal{U}) \subset \mathcal{S}$ is a semioval of size k . \square

Corollary 2.2. *Let \mathcal{B} be a 2-blocking set of \mathcal{H}_q . Then in $\text{PG}(2, q^2)$ there exist semiovals of size k for all k satisfying the inequalities $|\mathcal{B}| \leq k \leq q^3 + 1$.*

Proof. The set $\mathcal{T} = \mathcal{H}_q \setminus \mathcal{B}$ satisfies the condition of Proposition 2.1. \square

Our first, obvious construction does not depend on the parity of q .

Theorem 2.3. *Let $q \geq 2$. In $\text{PG}(2, q^2)$ there exists a semioval $\mathcal{S} \subset \mathcal{H}_q$ of size k for all $k \in \{q^3 - q^2 + q\} \cup [q^3 - q^2 + q + 2, q^3 + 1]$.*

Proof. Let P be a point in \mathcal{H}_q and $\ell_1, \ell_2, \dots, \ell_{q-1}$ be $(q+1)$ -secants through P . Let $\mathcal{T} = \bigcup_{i=1}^{q-1} \ell_i$. Then $\mathcal{H}_q \setminus \mathcal{T}$ is a semioval of size $q^3 - q^2 + q$ because if ℓ is a $(q+1)$ -secant of \mathcal{H}_q then we either deleted all of its points, or at most $q-1$ of its points. Hence no former secant line becomes a tangent line to $\mathcal{H}_q \setminus \mathcal{T}$, and there is exactly one tangent line at each point of $\mathcal{H}_q \setminus \mathcal{T}$, then $\mathcal{H}_q \setminus \mathcal{T}$ is a semioval of size $q^3 - q^2 + q$. Also, we can add $\bar{k} \in [2, q^2 - q + 1]$ points from \mathcal{T} in a way that no other tangent lines are created. In fact, it is sufficient to control that in each line $\ell_1, \ell_2, \dots, \ell_{q-1}$ the number of added points is different from 1. This is always possible since $\bar{k} \neq 1$. \square

The next two algebraic constructions work for all q , but the lower bound of the size depend on the parity of q . We use the following description of $\mathcal{H}_q \subset \text{PG}(2, q^2)$. The curve \mathcal{H}_q is defined by the equation

$$X_2 X_0^q + X_2^q X_0 + X_1^{q+1} = 0. \quad (1)$$

Let $c \in \mathbb{F}_{q^2}$ be a fixed root of the equation $c^q + c + 1 = 0$. Consider the set

$$M = \{m \in \mathbb{F}_{q^2} \mid m^q + m = 0\}, \quad (2)$$

then the points of \mathcal{H}_q are

$$\{(1 : u : cu^{q+1} + m) \mid u \in \mathbb{F}_{q^2}, m \in M\} \cup \{(0 : 0 : 1)\}. \quad (3)$$

If q is an odd prime power then let h be a fixed non-square in \mathbb{F}_q and consider $\mathbb{F}_{q^2} = \mathbb{F}_q[i]$ where $i^2 = h$. Then $i^q + i = 0$, $i^2 = i^{2q}$ and $i^{q+1} = -h$.

If q is even then let $h \in \mathbb{F}_q$ be an element with $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(h) = 1$. Let $i^2 + i + h = 0$, and consider $\mathbb{F}_{q^2} = \mathbb{F}_q[i]$. Then $i^q + i = 1$, $i^2 = i + h$ and $i^{q+1} = h$.

For all q we represent the elements x of \mathbb{F}_{q^2} as $x = x_1 + ix_2$ where $x_1, x_2 \in \mathbb{F}_q$.

Theorem 2.4. *If $q \geq 3$ then in $\text{PG}(2, q^2)$ there exists a semioval $\mathcal{S} \subset \mathcal{H}_q$ of size k for all k satisfying the inequalities*

$$q^3 + 1 \geq k \geq \begin{cases} q^3 - 2q^2 + 4q + 1 & \text{if } q \text{ is even,} \\ q^3 - 2q^2 + 5q - 2 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Take the subset

$$\mathcal{T} = \{(1 : u : cu^{q+1} + m) \mid u \in \mathbb{F}_q, m \in M, m \neq 0\} \quad (4)$$

of points of \mathcal{H}_q . Note that $|\mathcal{T}| = q(q-1)$. The condition $u \in \mathbb{F}_q$ implies that $u^{q+1} = u^2$, thus the points of \mathcal{T} can be written as $(1 : u : cu^2 + m)$, too. We claim that the set $\mathcal{U} = \mathcal{H}_q \setminus \mathcal{T}$ is a semioval. Because of Proposition 2.1 it is enough to prove that any line of $\text{PG}(2, q^2)$ contains at most $q-1$ points of \mathcal{T} .

First consider the lines through the point $P = (0 : 0 : 1)$. The line $X_0 = 0$ is the tangent to \mathcal{H}_q at P , while a line ℓ_α having equation $X_1 = \alpha X_0$ meets \mathcal{T} in points whose second coordinate is α . Thus ℓ_α contains $q-1$ points of \mathcal{T} if $\alpha \in \mathbb{F}_q$ and no points of \mathcal{T} if $\alpha \notin \mathbb{F}_q$.

Now consider the other lines of the plane. If a line ℓ does not contain P then its equation can be written as $\alpha X_0 + \beta X_1 + X_2 = 0$. If the point $(1 : u : cu^2 + m)$ is on ℓ then $\alpha + \beta u + cu^2 + m = 0$, hence we get

$$m = -cu^2 - \beta u - \alpha \quad \text{and} \quad m^q = -c^q u^2 - \beta^q u - \alpha^q.$$

But m satisfies the condition $m^q + m = 0$ hence

$$-(c^q + c)u^2 - (\beta^q + \beta)u - \alpha^q - \alpha = 0. \quad (5)$$

The coefficient of u^2 is 1 since $c^q + c + 1 = 0$. So Equation (5) is a quadratic equation on u , it has at most two roots. Hence ℓ contains at most 2 points of \mathcal{T} and $2 \leq q-1$ because $q > 2$.

In the second step we get smaller semiovals by careful deletion of points of \mathcal{U} .

Let $M = \{0, m_1, \dots, m_{q-1}\}$ and for $j = 1, 2, \dots, q-1$ let ℓ_j be the line with equation $X_2 = (c + m_j)X_0$. Then ℓ_j passes on the point $(0 : 1 : 0)$. The point $\ell_\alpha \cap \ell_j$ has coordinates $(1 : \alpha : c + m_j)$, thus it belongs to \mathcal{T} if and only if $\alpha \in \mathbb{F}_q$ and $\alpha^{q+1} = 1$ hold simultaneously. It happens if and only if $\alpha^2 = 1$, hence ℓ_j contains $q-1$ or q points of $\mathcal{H}_q \setminus \mathcal{T}$ if q is odd or even, respectively. If $q = 3$ then \mathcal{U} is a semioval of size $q^3 - q^2 + q + 1 = 22$. Let $q > 3$ and consider $\mathcal{V} = \bigcup_{j=1}^{q-3} \ell_j$. Then $\mathcal{S}_0 = \mathcal{U} \setminus \mathcal{V}$ is a semioval: in fact each $(q+1)$ -secant of \mathcal{H}_q contains at most two points of \mathcal{T} and $q-3$ points of \mathcal{V} hence it is not a tangent to \mathcal{S}_0 .

The size of \mathcal{S}_0 is $q^3 - q^2 + q + 1 - (q-1)(q-3) = q^3 - 2q^2 + 5q - 2$ if q is odd and $q^3 - q^2 + q + 1 - q(q-3) = q^3 - 2q^2 + 4q + 1$ if q is even. Note that, contrary to Theorem 2.3, we can add also one point to \mathcal{S}_0 : in fact it is sufficient to add a point in \mathcal{T} and the new set is still a semioval. \square

Theorem 2.5. *If $q > 2$ then in $\text{PG}(2, q^2)$ there exists a semioval $\mathcal{S} \subset \mathcal{H}_q$ of size k for all k satisfying the inequalities*

$$q^3 + 1 \geq k \geq \begin{cases} \frac{q^3 + 2q^2 - q + 2}{2} & \text{if } q \text{ is odd,} \\ \frac{q^3 + 3q^2 - 2q + 2}{2} & \text{if } q \text{ is even.} \end{cases}$$

Proof. Consider $\mathbb{F}_{q^2} = \mathbb{F}_q[i]$. Let $v \in \mathbb{F}_q$ be a fixed element. Take the subset

$$\mathcal{T}_v = \{(1 : u + iv : c(u + iv)^{q+1} + m) \mid u \in \mathbb{F}_q, m \in M, m \neq 0\} \quad (6)$$

of points of \mathcal{H}_q . Note that $|\mathcal{T}_v| = q(q-1)$.

If a line ℓ does not contain the point $P = (0 : 0 : 1)$ then its equation can be written as $\alpha X_0 + \beta X_1 + X_2 = 0$. We claim that ℓ contains at most two points of \mathcal{T}_v .

First consider the case q odd. The condition $u, v \in \mathbb{F}_q$ implies that $(u + iv)^{q+1} = u^2 - h^2 v^2$, thus the points of \mathcal{T}_v can be written as $(1 : u + iv : c(u^2 - h^2 v^2) + m)$, too. If the point $(1 : u : c(u^2 - h^2 v^2) + m)$ is on ℓ then $\alpha + \beta(u + iv) + c(u^2 - h^2 v^2) + m = 0$. Rearranging this we get

$$m = -cu^2 - \beta u - \alpha - \beta iv + ch^2 v^2 \quad \text{and} \quad m^q = -c^q u^2 - \beta^q u - \alpha^q + \beta^q iv + c^q h^2 v^2.$$

But m satisfies the condition $m^q + m = 0$ hence

$$-(c^q + c)u^2 - (\beta^q + \beta)u - \alpha^q - \alpha - (\beta + \beta^q)iv + (c^q + c)h^2 v^2 = 0. \quad (7)$$

If q is even then we get a similar equation. The condition $u, v \in \mathbb{F}_q$ now implies that $(u + iv)^{q+1} = u^2 + hv^2$, thus the points of \mathcal{T}_v can be written as $(1 : u + iv : c(u^2 + hv^2) + m)$. If the point $(1 : u : c(u^2 + hv^2) + m)$ is on ℓ then

$$m = cu^2 + \beta u + \alpha + \beta iv + chv^2 \quad \text{and} \quad m^q = c^q u^2 + \beta^q u + \alpha^q + \beta^q iv + \beta^q v + c^q hv^2.$$

But m satisfies the condition $m^q + m = 0$ hence

$$(c^q + c)u^2 + (\beta^q + \beta)u + \alpha^q + \alpha + (\beta + \beta^q)iv + \beta^q v + (c^q + c)hv^2 = 0. \quad (8)$$

The coefficient of u^2 in Equations (7) and (8) is 1 since $c^q + c + 1 = 0$. So these are quadratic equations on u , each of them has at most two roots. Hence ℓ contains at most 2 points of \mathcal{T}_v .

Let $v_1, v_2, \dots, v_{\lfloor (q-1)/2 \rfloor}$ be distinct elements of \mathbb{F}_q and let $\mathcal{T} = \bigcup_{i=j}^{\lfloor (q-1)/2 \rfloor} \mathcal{T}_{v_j}$. We show that $S_0 = \mathcal{H}_q \setminus \mathcal{T}$ is a semioval.

Because of Proposition 2.1 it is enough to prove that any $(q+1)$ -secant of \mathcal{H}_q contains at most $q-1$ points of \mathcal{T} . This is obvious if a line does not contain the point P because in this case it contains at most two points from each set \mathcal{T}_{v_j} . Consider the lines through the point $P = (0 : 0 : 1)$. The line $X_0 = 0$ is the tangent to \mathcal{H}_q at P , while a line ℓ_α having equation $X_1 = \alpha X_0$ meets \mathcal{T}_{v_j} in points whose second coordinate is α . Thus ℓ_α contains $q-1$ points of \mathcal{T}_{v_j} if $(\alpha - iv_j) \in \mathbb{F}_q$ and no points of \mathcal{T}_{v_j} if $(\alpha - iv_j) \notin \mathbb{F}_q$.

The size of S_0 is $q^3 + 1 - q(q-1)^2/2$ if q is odd and $q^3 + 1 - q(q-1)(q-2)/2$ if q is even. Thus the theorem follows from Proposition 2.1. □

If $q - 1$ has suitable divisors then we can construct smaller semiovals than in Theorem 2.4. We distinguish the cases q even and q odd.

First let q be an odd prime power. The following lemma due to Blokhuis et al. [6] gives information on the irreducibility of a particular plane curve.

Lemma 2.6 ([6], Lemma 4.5). *Let q be odd. If $n_1^2 \neq 2d_1 + hn_2^2$ then the algebraic curve in $\text{PG}(2, q)$ defined as*

$$\mathcal{X}_{\text{odd}} : 2n_1X_0^rX_2^r + 2hn_2X_1X_2^{2r-1} + 2d_1X_2^{2r} + X_0^{2r} - hX_1^2X_2^{2r-2} = 0 \quad (9)$$

is absolutely irreducible and it has genus $g = r - 1$.

If q is even then a similar lemma holds.

Lemma 2.7 ([6], page 14). *Let q be even. If $n_1^2 \neq n_1n_2 + hn_2^2 + d_2 \neq 0$ then the algebraic curve in $\text{PG}(2, q)$ defined as*

$$\mathcal{X}_{\text{even}} : (n_1 + n_2)X_1X_2^{2r-1} + n_2X_0^rX_2^r + d_2X_2^{2r} + X_0^{2r} + X_0^rX_1X_2^{r-1} + hX_1^2X_2^{2r-2} = 0 \quad (10)$$

is absolutely irreducible and it has genus $g \leq r - 1$.

Using these lemmas, Blokhuis et al. [6] constructed a blocking set of \mathcal{H}_q . With a slight modification of their proof we can prove the existence of a family of semiovals contained in the Hermitian curve.

Theorem 2.8. *Let q be a prime power and r be a divisor of $q - 1$ for which $r < \frac{\sqrt{q}}{2}$ holds. Then in $\text{PG}(2, q^2)$ there exists a semioval $\mathcal{S} \subset \mathcal{H}_q$ of size k for all k satisfying the inequalities*

$$\left(q + \frac{(q-1)q}{r} \right) (q-1) + q^2 + 1 \leq k \leq q^3 + 1. \quad (11)$$

Proof. The equation of \mathcal{H}_q is the same

$$X_2X_0^q + X_2^qX_0 + X_1^{q+1} = 0$$

as in the previous proof, but we use another description of its points.

The point $X_\infty = (1 : 0 : 0)$ is on \mathcal{H}_q and the line $\ell_\infty : X_2 = 0$ is the tangent to \mathcal{H}_q at X_∞ . Choose ℓ_∞ as line at infinity and consider $\text{PG}(2, q^2)$ as the union of $\text{AG}(2, q^2)$ and ℓ_∞ . Let \mathcal{U} be the affine part of \mathcal{H}_q . Then the equation of \mathcal{U} is

$$X^q + X + Y^{q+1} = 0. \quad (12)$$

The tangent to \mathcal{H}_q at the affine point (a, b) has equation $X = -b^q Y - a^q$. Thus a non-horizontal affine line $X = nY + d$ is a tangent to \mathcal{H}_q if and only if $n^{q+1} \neq d^q + d$.

Take the following pointset

$$\mathcal{B} = \{(x, y) \in \mathcal{U} \mid y = u^r + iv, u, v \in \mathbb{F}_q\} \cup \{(1 : 0 : 0)\} \subset \mathcal{H}_q. \quad (13)$$

For all $u, v \in \mathbb{F}_q$ the horizontal affine line with equation $Y = u^r + iv$ contains q affine points of \mathcal{B} , the other horizontal lines $Y = y_0 \neq y = u^r + iv$ does not contain any affine point of \mathcal{B} . There is exactly one point of the line ℓ_∞ in \mathcal{B} . Thus \mathcal{B} consists of $\left(q + \frac{(q-1)q}{r}\right)q + 1$ points.

First consider the case q odd. Let ℓ be a non-horizontal affine line with equation $X = nY + d$. Let $n = n_1 + in_2$ and $d = d_1 + id_2$ where $n_1, n_2, d_1, d_2 \in \mathbb{F}_q$. Then

$$\ell \cap \mathcal{B} = \{(x, y) \in \mathcal{U} \mid y = u^r + iv, 2n_1u^r + 2hn_2v + 2c_1 + u^{2r} - hv^2 = 0, u, v \in \mathbb{F}_q\}.$$

Thus affine points $(x, u^r + iv)$ of \mathcal{B} on the line ℓ correspond to points of the curve having affine equation

$$\mathcal{A}_{\text{odd}} : 2n_1U^r + 2hn_2V + 2d_1 + U^{2r} - hV^2 = 0.$$

This curve is the affine part of the curve \mathcal{X}_{odd} in $\text{PG}(2, q)$. Suppose that ℓ is not a tangent line to \mathcal{U} . Then $n^{q+1} \neq d^q + d$. It holds if and only if $n_1^2 \neq 2d_1 + hn_2^2$. So in this case by Lemma 2.6, \mathcal{X}_{odd} is absolutely irreducible and its genus is equal to $r - 1$.

If q is even and ℓ is a non-horizontal affine line with equation $X = nY + d$, $n = n_1 + in_2$ and $d = d_1 + id_2$ where $n_1, n_2, d_1, d_2 \in \mathbb{F}_q$ then

$$\ell \cap \mathcal{B} = \{(x, y) \in \mathcal{U} \mid y = u^r + iv, n_2u^r + (n_1 + n_2)v + d_2 + u^{2r} + u^rv + hv^2 = 0, u, v \in \mathbb{F}_q\}.$$

Thus affine points $(x, u^r + iv)$ of \mathcal{B} on the line ℓ correspond to points of the curve having affine equation

$$\mathcal{A}_{\text{even}} : n_2U^r + (n_1 + n_2)V + d_2 + U^{2r} + U^rV + hV^2 = 0.$$

This curve is the affine part of the curve $\mathcal{X}_{\text{even}}$ in $\text{PG}(2, q)$. Suppose that ℓ is not a tangent line to \mathcal{U} . Then $n^{q+1} \neq d^q + d$. It holds if and only if $n_1^2 \neq n_1n_2 + hn_2^2 + d_2$. So in this case by Lemma 2.7, $\mathcal{X}_{\text{even}}$ is absolutely irreducible and its genus is at most to $r - 1$.

For all q , the Hasse-Weil bound implies that each of the curves \mathcal{X}_{odd} and $\mathcal{X}_{\text{even}}$ has at least $q + 1 - 2(r - 1)\sqrt{q}$ points in $\text{PG}(2, q)$. Both curves have a unique point at infinity, $(0 : 1 : 0)$. Now consider the points of the curves \mathcal{A}_{odd} and $\mathcal{A}_{\text{even}}$. The line $U = 0$ contains at most two points of these curves. If $u \neq 0$, ξ is an r -th root of the unity and (u, v) is on \mathcal{A}_{odd} or on $\mathcal{A}_{\text{even}}$ then the point $(\xi u, v)$ is also on \mathcal{A}_{odd} or on $\mathcal{A}_{\text{even}}$, respectively. But if ϵ is a primitive r -th root of the unity,

then for $i = 0, 1, \dots, r-1$ the points (u, v) and $(\epsilon^i u, v)$ of the curves give the same affine point $(x, y) = (x, u^r + iv)$ of $\ell \cap \mathcal{B}$. Hence $\ell \cap \mathcal{B}$ contains at least

$$\frac{(q+1 - (2r-2)\sqrt{q}) - 1}{r}$$

points, and by the assumption on r this number is greater than 2. Thus the set $\ell \cap \mathcal{B}$ contains at least two points.

All horizontal lines pass through $(0 : 1 : 0) \in \mathcal{B}$. Since no horizontal line is a tangent line to \mathcal{U} , it is sufficient to add one point for each of the lines with equation $Y = y_0 \neq y = u^r + iv$ to extend \mathcal{B} to a 2-blocking set of \mathcal{H}_q . Let \mathcal{S}_0 be the set obtained from \mathcal{B} by adding these extra points. Then the size of \mathcal{S}_0 is

$$\underbrace{\left(q + \frac{(q-1)q}{r}\right)}_{|\mathcal{B}|} q + 1 + \underbrace{\left[q^2 - \left(q + \frac{(q-1)q}{r}\right)\right]}_{\text{extra points on horizontal lines}} = \left(q + \frac{(q-1)q}{r}\right) (q-1) + q^2 + 1.$$

Now the theorem follows from Corollary 2.2. \square

The lower bound on the size of the semioval in Theorem 2.8 depends on the divisor r of $q-1$. The greater r the smaller the size of \mathcal{S}_0 , but the method works only if $r < \sqrt{q}/2$ holds. Note that if we take $r = 1$ in Theorem 2.8 then the semioval obtained is just the Hermitian curve itself. If $1 < r$ then the condition $r < \sqrt{q}/2$ implies $q > 16$.

Also, if q is even or odd, then sometimes $q-1 = p$ or $q-1 = 2p$, respectively, where p is a prime number. Hence there is no $r \neq 1$ (e.g. in the cases $q = 32, 128$) or the best possible value is $r = 2$ (e.g. in the cases $q = 23, 27$). In the case $r = 2$ Theorem 2.8 gives semiovals of sizes

$$\frac{q^3 + 2q^2 - q + 2}{2} \leq k \leq q^3 + 1.$$

This is the same as the result of Theorem 2.5.

The situation is much better if q is an odd square. In the case $q = s^2$, s odd, one can always choose $r = (s-1)/2$: this is the greatest possible divisor of $q-1$ satisfying the condition $r < \sqrt{q}/2$. In this case Theorem 2.8 has the following

Corollary 2.9. *Let q be an odd square. Then in $\text{PG}(2, q^2)$ there exists a semioval $\mathcal{S} \subset \mathcal{H}_q$ of size k for all k satisfying the inequalities*

$$2q^2 \sqrt{q} + 4q^2 - 2q\sqrt{q} - 3q + 1 \leq k \leq q^3 + 1.$$

If $q \geq 49$, then this extends the spectrum of sizes of constructed semiovals in $\text{PG}(2, q^2)$ significantly, because previously the corresponding lower bound was $(q^3 + q^2)/2$ (see [21]).

If $q = s^t$, s odd and $t > 2$ then $r = s - 1$ is always a possible choice. In this case the size of the smallest semioval we get is roughly $q^2 \sqrt[t]{q^{t-1}}$.

3 The proof of existence of smaller semiovals

If q is odd then we can prove the existence of much smaller semiovals using a theorem about dominating sets of bipartite graphs. Let A and B be the two vertex subsets of a bipartite graph. We say that a vertex $v \in B$ dominates the subset $S \subset A$, if for any $s \in S$ there is an edge between v and s . A subset $B' \subset B$ is a dominating set, if for any $a \in A$ there exists $b' \in B'$ which dominates a . The following lemma is due to S. K. Stein, the proof can be found e.g. in [15].

Lemma 3.1. *Let A and B be the two vertex subsets of a bipartite graph. Denote by d the minimum degree in A . If A has at least two elements, then there is a set $B' \subset B$ dominating the vertices of A with*

$$|B'| \leq \left\lceil |B| \frac{\log(|A|)}{d} \right\rceil,$$

where \log denotes natural base logarithm.

Theorem 3.2. *Let $q > 27$ be odd and k be an integer satisfying*

$$(q^2 - q + 1) \left\lceil \frac{8(q+1)}{q-1} \log q \right\rceil \leq k \leq q^3 + 1,$$

where \log denotes the natural base logarithm. Then $\text{PG}(2, q^2)$ contains semiovals of size k .

Proof. The Hermitian curve \mathcal{H}_q is the disjoint union of $q+1$ $(q^2 - q + 1)$ -arcs. Let $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{q+1}\}$ denote the set of these arcs. If P is a point of \mathcal{H}_q that belongs to \mathcal{C}_i , then the set of the $q+1$ tangents to \mathcal{C}_i at P contains the unique tangent to \mathcal{H}_q at P , while each of the remaining q lines is a tangent to exactly one another element of \mathcal{C} ; see [6, page 10]. Thus any bisecant of \mathcal{C}_i is a tangent to either 0 or 2 other arcs from \mathcal{C} , hence is a bisecant of $(q-1)/2$ or $(q+1)/2$ elements of \mathcal{C} .

We define a bipartite graph with two vertex subsets A and B . Let the vertices in B be the $(q^2 - q + 1)$ -arcs giving the decomposition of \mathcal{H} , and the vertices in A be those lines that are not tangents to \mathcal{H} . Let $a \in A$ and $b \in B$ be joined if and only if the corresponding line is a bisecant of

the corresponding arc. Then $|B| = q + 1$, $|A| = q^4 - q^3 + q^2$ and $d = (q - 1)/2$. Hence from Lemma 3.1 we get that there exists $B' \subset B$ dominating A , and

$$|B'| \leq \left\lceil |B| \frac{\log(|A|)}{d} \right\rceil = \left\lceil (q + 1) \frac{\log(q^4 - q^3 + q^2)}{(q - 1)/2} \right\rceil \leq \left\lceil \frac{8(q + 1)}{q - 1} \log q \right\rceil.$$

Thus there exists a subset of $\left\lceil \frac{8(q+1)}{q-1} \log q \right\rceil$ arcs $\mathcal{V} \subset \mathcal{U}$ such that each secant of \mathcal{H} meets \mathcal{V} in at least two points. Hence \mathcal{V} is a semioval of size

$$k_0 = (q^2 - q + 1) \left\lceil \frac{8(q + 1)}{q - 1} \log q \right\rceil. \quad (14)$$

If $k_0 < k \leq q^3 + 1$ then Corollary 2.2 guarantees the existence of a semioval of size k . \square

In [21] it was proved that there exist semiovals of sizes greater than

$$q^2 \left\lceil \frac{8q}{q + 1} \log q \right\rceil + 1. \quad (15)$$

The bound (14) is smaller than the bound (15) for infinitely many q . In fact, this happens when

$$\left\lceil \frac{8q}{q + 1} \log q \right\rceil = \left\lceil \frac{8(q + 1)}{q - 1} \log q \right\rceil.$$

Let $e = \lfloor \log q \rfloor$ and $f = \log q - \lfloor \log q \rfloor$. The previous equality is satisfied whenever

$$\frac{e}{q} \leq f < \frac{q - 1}{8(q + 1)} - \frac{2e}{q + 1}.$$

If q is large enough, then $\frac{e}{q} = \epsilon$, with ϵ close to zero, and $\frac{q-1}{8(q+1)} - \frac{2e}{q+1}$ is greater than $1/9$. Therefore, for all q such that

$$\epsilon \leq \log q - \lfloor \log q \rfloor < \frac{1}{9}$$

our bound is better than the previously known (the smallest prime power satisfying this condition is $q = 137$).

4 2-blocking sets of the Hermitian curve

In this section we present a construction of an infinite family of 2-blocking sets of the Hermitian curve which is a modification of the construction of 1-blocking sets of the Hermitian curve presented in [6]. Here we use the description of the Hermitian curve as given in (12).

Proposition 4.1. *Let q be an odd prime power and r be a divisor of $q - 1$ such that $1 < r < \frac{\sqrt{q}}{2} - \frac{2\log_2 q + 1}{4}$. Then there exists a 2-blocking set of the Hermitian curve for some value of k satisfying the inequalities*

$$\frac{q^3 - 3q^2 - 2q}{r} + 2q^2 - q + 2 - 2\lceil \log_2 q + 1 \rceil \leq k \leq \frac{q^3 - 3q^2 - 2q}{r} + 2q^2 + q + 2 + 2\lceil \log_2 q + 1 \rceil.$$

Proof. Let \mathcal{B} defined by

$$\mathcal{B} = \{(x, y) \in \mathcal{U} \mid y = u^r + iv, u, v \in \mathbb{F}_q\} \cup \{(1 : 0 : 0)\} \subset \mathcal{H}_q.$$

As in Theorem 2.8 it is possible to prove that each non-horizontal line, which is not a tangent line of the Hermitian curve, intersects \mathcal{B} in at least $\frac{q-1-(2r-2)\sqrt{q}}{r}$ and in at most $\frac{q-1+(2r-2)\sqrt{q}}{r}$ points. The horizontal lines are either blocked only by $(1 : 0 : 0)$ or they are fully contained in \mathcal{B} . Consider a $(q^2 - q + 1)$ -arc \mathcal{C} through $(1 : 0 : 0)$ contained in \mathcal{U} . Among the q^2 horizontal lines through $(1 : 0 : 0)$, q of them intersect \mathcal{C} only in $(1 : 0 : 0)$, while the other $q^2 - q$ contain an extra point of \mathcal{C} other than $(1 : 0 : 0)$. Consider

$$\overline{\mathcal{B}} := (\mathcal{B} \Delta \mathcal{C}) \cup \{(1 : 0 : 0)\},$$

where Δ indicates the symmetric difference of the two sets. The set $\overline{\mathcal{B}}$ still blocks all the non-horizontal lines and they are not completely contained in $\overline{\mathcal{B}}$, since on each of these lines at most two points are deleted from \mathcal{B} or added to \mathcal{B} . Also, at most q horizontal lines, namely the q unisecant lines to \mathcal{C} through $(1 : 0 : 0)$, either intersect $\overline{\mathcal{B}}$ only in $(1 : 0 : 0)$ or they are fully contained in $\overline{\mathcal{B}}$.

Consider $\mathcal{C}_1, \dots, \mathcal{C}_{q+1}$ the $(q^2 - q + 1)$ -arcs partitioning \mathcal{U} and let ℓ_1, \dots, ℓ_k be $k \leq q$ horizontal lines. There exists at least a $(q^2 - q + 1)$ -arc \mathcal{C}_i contained in \mathcal{U} intersecting at least $\frac{k}{2}$ of such lines. On the contrary, suppose that all the $(q^2 - q + 1)$ -arcs contained in \mathcal{U} intersect at most $\frac{k}{2} - 1$ of such lines. Then, each arc contains at most $k - 2$ points of $\mathcal{U} \cap (\ell_1 \cup \dots \cup \ell_k)$. Since $\mathcal{U} \cap (\ell_1 \cup \dots \cup \ell_k) = kq + 1$, then there should exist at least $\frac{kq+1}{k-2} > q + 1$ of such arcs contained in \mathcal{U} and pairwise disjoint. This is a contradiction.

Arguing as before we can prove that, given k horizontal lines, there exist at most $\lceil \log_2 k + 1 \rceil$ arcs among $\mathcal{C}_1, \dots, \mathcal{C}_{q+1}$ such that their union intersects all the k lines. Note that in this case each of these lines contains at most $2\lceil \log_2 k + 1 \rceil$ points from the union of the $(q^2 - q + 1)$ -arcs.

Let $\ell_1, \dots, \ell_{k_1}$ and r_1, \dots, r_{k_2} be the horizontal lines intersecting $\overline{\mathcal{B}}$ only in $(1 : 0 : 0)$ and in $q+1$ points, respectively. From above we know that $k_1 + k_2 \leq q$. Let $\mathcal{C}_{i_1}, \dots, \mathcal{C}_{i_{j_1}}$, $j_1 = \lceil \log_2 k_1 + 1 \rceil$, be the $(q^2 - q + 1)$ -arcs intersecting $\ell_1, \dots, \ell_{k_1}$ and let $\mathcal{C}_{h_1}, \dots, \mathcal{C}_{h_{j_2}}$, $j_2 = \lceil \log_2 k_2 + 1 \rceil$, be the $(q^2 - q + 1)$ -arcs intersecting r_1, \dots, r_{k_2} . In particular, let

$$\mathcal{B}_1 := (\ell_1 \cup \dots \cup \ell_{k_1}) \cap (\mathcal{C}_{i_1}, \dots, \mathcal{C}_{i_{j_1}})$$

and

$$\mathcal{B}_2 := (r_1 \cup \dots \cup r_{k_2}) \cap (\mathcal{C}_{h_1}, \dots, \mathcal{C}_{h_{j_2}}).$$

We have that $|\mathcal{B}_1| \leq 2\lceil \log_2 k_1 + 1 \rceil$ and $|\mathcal{B}_2| \leq 2\lceil \log_2 k_2 + 1 \rceil$. The set

$$\tilde{\mathcal{B}} = (\overline{\mathcal{B}} \setminus \mathcal{B}_2) \cup \mathcal{B}_1 \cup \{(1 : 0 : 0)\}$$

intersects each horizontal line in at least two and in at most q points. Also, since a non-horizontal line ℓ intersects $\overline{\mathcal{B}}$ in t points, where

$$2\lceil \log_2 q + 1 \rceil + 2 < \frac{q-1-(2r-2)\sqrt{q}}{r} \leq t \leq \frac{q-1+(2r-2)\sqrt{q}}{r} < q - 2\lceil \log_2 q + 1 \rceil,$$

then ℓ intersects $\tilde{\mathcal{B}}$ in \tilde{t} points, where

$$2 < \frac{q-1-(2r-2)\sqrt{q}}{r} \leq \tilde{t} \leq \frac{q-1+(2r-2)\sqrt{q}}{r} < q.$$

This proves that $\tilde{\mathcal{B}}$ is a 2-blocking set of the Hermitian curve not containing any block of it.

Finally, note that the size of \mathcal{B} is $\left(q + \frac{(q-1)q}{r}\right)q + 1$ and that the points of \mathcal{B} lie on $s = q + \frac{q(q+1)}{r}$ horizontal lines. Therefore the size of $\overline{\mathcal{B}}$ satisfies

$$\left(q + \frac{(q-1)q}{r}\right)q + 1 + (q^2 - q - 2s + 1) \leq |\overline{\mathcal{B}}| \leq \left(q + \frac{(q-1)q}{r}\right)q + 1 + (q^2 + q - 2s + 1).$$

To obtain $\tilde{\mathcal{B}}$ we add or delete at most $2\lceil \log_2 q + 1 \rceil$ points. So,

$$\begin{aligned} \left(q + \frac{(q-1)q}{r}\right)q + 1 + (q^2 - q - 2s + 1) - 2\lceil \log_2 q + 1 \rceil &\leq |\tilde{\mathcal{B}}| \leq \\ &\leq \left(q + \frac{(q-1)q}{r}\right)q + 1 + (q^2 + q - 2s + 1) + 2\lceil \log_2 q + 1 \rceil. \end{aligned}$$

□

References

- [1] D. Bartoli, A.A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, On sizes of complete arcs in $\text{PG}(2, q)$, *Discrete Math.* **312** (2012), 680–698.
- [2] D. Bartoli, On the structure of semiovals of small size, *J. Combin. Des.*, DOI: 10.1002/jcd.21383.
- [3] D. Bartoli, S. Marcugini, and F. Pambianco, The non-existence of some NMDS codes and the extremal sizes of complete $(n, 3)$ arcs in $\text{PG}(2, 16)$, *Des. Codes Cryptogr.* **72**(1) (2014), 129–134.
- [4] L.M. Batten, Determining sets, *Australas. J. Combin.* **22** (2000), 167–176.
- [5] J. Bierbrauer and Y. Edel, 41 is the largest size of a cap in $\text{PG}(4, 4)$, *Des. Codes Cryptogr.* **16** (1999), 151–160.
- [6] A. Blokhuis, D. Jungnickel, V. Krčadinac, S. Rottey, L. Storme, T. Szőnyi, and P. Vanden-driessche, Blocking sets of the classical unital, *Finite Fields Appl.* **35** (2015), 1–15.
- [7] K. Coolsaet and H. Sticker, The complete $(k, 3)$ -arcs of $\text{PG}(2, q)$, $q \leq 13$, *J. Combin. Des.* **20** (2012), 89–111.
- [8] B. Csajbók and Gy. Kiss, Notes on semiarcs, *Mediterr. J. Math.* **9** (2012), 677–692.
- [9] A.A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, On sizes of complete caps in projective spaces $\text{PG}(n, q)$ and arcs in planes $\text{PG}(2, q)$, *J. Geom.* **94** (2009), 31–58.
- [10] J.M. Dover, A lower bound on blocking semiovals, *European J. Combin.* **21** (2000), 571–577.
- [11] J.M. Dover, Some new results on small blocking semiovals, *Austral. J. Combin.* **52** (2012), 269–280.
- [12] J.M. Dover and K.E. Mellinger, Semiovals from unions of conics, *Innov. Incidence Geom.* **12** (2011), 61–83.
- [13] V. Fack, S.L. Fancsali, L. Storme, G. Van de Voorde, and J. Winne, Small weight codewords in the codes arising from Desarguesian projective planes, *Des. Codes Cryptogr.* **46** (2008), 25–43.
- [14] J.C. Fisher, J.W.P. Hirschfeld, and J.A. Thas, Complete arcs in planes of square order, *Ann. Discrete Math.* **30** (1986), 243–250.

- [15] A. Gács and T. Szőnyi, Random constructions and density results, *Des. Codes Cryptogr.* **47** (2007), 267–287.
- [16] J. W. P. Hirschfeld, *Projective geometries over finite fields*, 2nd edn. Clarendon Press, Oxford (1998).
- [17] J. W. P. Hirschfeld, G. Korchmáros, and F. Torres, *Algebraic Curves Over Finite Fields*, Princeton University Press, Princeton and Oxford (2008).
- [18] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory, and finite projective spaces, *Proceedings of the Fourth Isle of Thorns Conference*, Eds. A. Blokhuis, J. W. P. Hirschfeld, D. Jungnickel and J. A. Thas, Kluwer, (2001), 201–246.
- [19] X. Hubaut, Limitation du nombre de points d’un (k, n) -arc régulier d’un plan projectif fini, *Atti. Accad. Naz. Lincei Rend.* **8** (1970), 490–493.
- [20] Gy. Kiss, A survey on semiovals, *Contrib. Discrete Math.* **3** (2008), 81–95.
- [21] Gy. Kiss, S. Marcugini, and F. Pambianco, On the spectrum of the sizes of semiovals in $\text{PG}(2, q)$, q odd, *Discrete Math.* **310** (2010), 3188–3193.
- [22] P. Lisonek, Computer-assisted Studies in Algebraic Combinatorics, *Ph.D. Thesis*, RISC, J. Kepler University Linz, (1994).
- [23] S. Marcugini, A. Milani, and F. Pambianco, Maximal $(n, 3)$ -arcs in $\text{PG}(2, 13)$, *Discrete Math.* **294** (2005), 139–145.
- [24] S. Marcugini, A. Milani, and F. Pambianco, Complete arcs in $\text{PG}(2, 25)$: the spectrum of the sizes and the classification of the smallest complete arcs, *Discrete Math.* **307** (2007), 739–747.
- [25] F. Pambianco and L. Storme, Minimal blocking sets in $\text{PG}(2, 9)$, *Ars Combin.* **89** (2008), 223–234.
- [26] M. Seib, Unitäre Polaritäten endlicher projectiver Ebenen, *Arch. Math.* **21** (1970), 103–112.
- [27] T. Szőnyi, Note on the existence of large minimal blocking sets in Galois planes, *Combinatorica* **12** (1992), 227–235.
- [28] J. A. Thas, On semiovals and semiovoids, *Geom. Dedicata* **3** (1974), 229–231.

Daniele Bartoli
Department of Mathematics,
Ghent University,
Krijgslaan 281, 9000 Ghent, Belgium
e-mails: dbartoli@cage.ugent.be

György Kiss
Department of Geometry and MTA-ELTE GAC Research Group
Eötvös Loránd University
1117 Budapest, Pázmány s. 1/c, Hungary
e-mail: kissgy@cs.elte.hu

Stefano Marcugini and Fernanda Pambianco
Dipartimento di Matematica e Informatica, Università degli Studi di Perugia
Via Vanvitelli 1, 06123 Perugia, Italy
e-mails: gino@dmf.unipg.it, fernanda@dmf.unipg.it